

ON THE CONGRUENCE KERNEL OF ISOTROPIC GROUPS OVER RINGS

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1. INTRODUCTION

The aim of this text is to show that the theorem of A. Rapinchuk and I. Rapinchuk [RR] on the centrality of the elementary congruence kernel of a Chevalley group extends to the case of an isotropic simple simply connected group of isotropic rank ≥ 2 . We prove the following

Theorem 1. *Let G be a simple simply connected group scheme defined over a connected commutative ring A , of constant type Φ such that the structure constants of Φ are invertible in A . Assume moreover that the isotropic rank of G over A is ≥ 2 , that is, G contains at least two distinct parabolic subgroups $P_1 < P_2 < G$ over R . Let R be a Noetherian commutative A -algebra, and let $E(R)$ be the elementary subgroup of $G(R)$. We denote by $\widehat{E(R)}$ and $\overline{E(R)}$ the profinite and congruence completions of $E(R)$ respectively. Then the kernel $\mathcal{C}_E(R)$ of the natural map $\widehat{E(R)} \rightarrow \overline{E(R)}$ is central in $\widehat{E(R)}$.*

Remark. Recently M. Ershov, A. Jaikin-Zapirain, M. Kassabov, and N. Nikolov, e.g. [EJZK] obtained several results establishing Kazhdan's property T for groups graded by root systems and satisfying some additional properties. Their results seem to imply our theorem (modulo some of our lemmas showing that the root grading is "strong") when R is a finitely generated ring and the system of relative roots of G with respect to a parabolic subgroup (see §2) is isomorphic to a root system in the usual sense. However, in general a system of relative roots is not a root system and does not have an associated Coxeter group.

For certain rings R , it is known that $G(R) = E(R)$, which implies that $\mathcal{C}_E(R)$ is in fact the full congruence kernel of $G(R)$. If G is a Chevalley group, this is true for any semi-local ring R . Combining the above theorem with the results of [St12], we obtain the following result for quasi-split groups.

Corollary 1. *Under the hypothesis of Theorem 1, assume moreover that $R = \mathbb{F}_q$ is a finite field (in particular, G is quasi-split), and A is a semilocal regular ring containing \mathbb{F}_q . For any $m, n \geq 0$, the congruence kernel of*

$$G(A[X_1, \dots, X_n]) \quad \text{and} \quad G(\mathbb{F}_q[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}])$$

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is central.

The proof of Theorem 1 relies on the following two theorems. In the statements Φ_P denotes the system of relative roots corresponding to a parabolic subgroup P , and $X_\alpha(V_\alpha)$, $\alpha \in \Phi_P$, are the corresponding relative root subschemes; see § 2 for the definitions. The elementary subgroup

$$E_P(R) = \langle X_\alpha(V_\alpha), \quad \alpha \in \Phi_P \rangle$$

is independent of the choice of a parabolic subgroup P when the isotropic rank of G is ≥ 2 by the main result of [PSt1].

Theorem 2. *Let R be a commutative ring, let G be a simple algebraic group of constant type Φ over R such that the structure constants of the root system Φ are invertible in R . Let P be a parabolic subgroup of G such that $\text{rank } \Phi_P \geq 2$ everywhere on $\text{Spec } R$, and let $E(R) = E_P(R)$ be the elementary subgroup of $G(R)$. For any normal subgroup $N \leq E(R)$ there exists an ideal $I = I(N)$ in R , such that $N \cap X_\alpha(V_\alpha) = X_\alpha(IV_\alpha)$ for any $\alpha \in \Phi_P$.*

Corollary 2. *Under the hypotheses of Theorem 2, assume that N is a normal subgroup of finite index in $E(R)$. Then I is an ideal of finite index in R . Consequently, the profinite and congruence topologies on $E(R)$ induce the same topology on $X_\alpha(V_\alpha)$ for all $\alpha \in \Phi_P$.*

Proof. Clear. □

Theorem 3. *Let R be a local ring, G be an isotropic simply connected simple reductive group over R , P a parabolic subgroup of G . We assume that $\text{rank } \Phi_P \geq 2$. Then the kernel of $s_P : St_P(R) \rightarrow E(R)$ is central in $St_P(R)$, where $St_P(R)$ is the natural Steinberg group associated to P over R (see Definition 5 in §4.1).*

The latter theorem is actually applied in the proof of Theorem 1 only to quasi-split groups G , in which case it was earlier claimed in [Abe2]. However, the proof in that paper is mostly missing, and the definition of the Steinberg group is more complicated and not so well-adapted to our purposes. We give an independent argument, which turned out to extend easily to all isotropic groups. It relies on the results of V. Deodhar in the field case [Deo].

2. PRELIMINARIES

The results of [PSt1] were obtained using the calculus of relative roots and relative roots subschemes of isotropic reductive groups. Here we recall their main properties proved in [PSt1, LSt]. They will be freely used in the rest of the paper.

Let A be a commutative ring. Let G be an isotropic reductive group scheme over A . Let $P = P^+$ and P^- be two opposite strictly proper parabolic A -subgroups of G , with the common Levi subgroup $L_P = P \cap P^-$. Relative root subschemes of G with respect to $P = P^+$, actually, depend on the choice of P^- or L_P , but their essential properties stay the same, so we will usually omit P^- from the notation.

It was shown in [PSt1] that we can represent $\text{Spec } A$ as a finite disjoint union

$$\text{Spec } A = \coprod_{i=1}^m \text{Spec } A_i,$$

so that the following conditions hold for $i = 1, \dots, m$:

- the root system of $G_{\overline{k(s)}}$ is the same for all $s \in \text{Spec } A_i$;
- the type of the parabolic subgroup $P_{\overline{k(s)}}$ of $G_{\overline{k(s)}}$ is the same for all $s \in \text{Spec } A_i$;
- if S_i/A_i is a Galois extension of rings such that G_{S_i} is of inner type, then for any $s \in \text{Spec } A_i$ the Galois group $\text{Gal}(S_i/A_i)$ acts on the Dynkin diagram D_i of $G_{\overline{k(s)}}$ via the same subgroup of $\text{Aut}(D_i)$.

The relative roots and relative root subschemes of G are correctly defined over each A_i , $1 \leq i \leq m$, that is, only locally in the Zariski topology on $\text{Spec } A$. However, since

$$E_P(A) = \langle U_P(A), U_{P^-}(A) \rangle = \prod_{i=1}^m E_P(A_i),$$

all properties of the elementary subgroup proved using relative root subschemes are usually easy to extend to the general case. From here until the end of this section, assume that $A = A_i$ for some i .

Denote by Φ the root system of G , by Π a set of simple roots of Φ , by D the corresponding Dynkin diagram. Then the $*$ -action on D is determined by a subgroup Γ of $\text{Aut } D$. Let J be the subset of Π such that $\Pi \setminus J$ is the type of $P_{\overline{k(s)}}$ (that is, the set of simple roots of the Levi subgroup $L_{\overline{k(s)}}$). Then J is Γ -invariant. Consider the projection

$$\pi = \pi_{J,\Gamma}: \mathbb{Z}\Phi \longrightarrow \mathbb{Z}\Phi / \langle \Pi \setminus J; \alpha - \sigma(\alpha), \alpha \in J, \sigma \in \Gamma \rangle.$$

Definition 1. *The set $\Phi_P = \pi(\Phi) \setminus \{0\}$ is called the system of relative roots with respect to the parabolic subgroup P . The rank of Φ_P is the rank of $\pi(\mathbb{Z}\Phi)$ as a free abelian group.*

If A is a local ring and P is a minimal parabolic subgroup of G , then Φ_P can be identified with the relative root system of G in the sense of [SGA3, Exp. XXVI §7] or [BT] in the field case.

In [PSt1], we associated to any relative root $\alpha \in \Phi_P$ a finitely generated projective A -module V_α and a closed embedding

$$X_\alpha: W(V_\alpha) \rightarrow G,$$

where $W(V_\alpha)$ is the affine group scheme over A defined by V_α , which is called a *relative root subscheme* of G . These subschemes possess several nice properties similar to that of elementary root subgroups of a split group, which we summarize below.

Theorem 4. [PSt1, Theorem 2, Lemma 9] *Let $\alpha \in \Phi_P$.*

(i) *There exist degree i homogeneous polynomial maps $q_\alpha^i: V_\alpha \oplus V_\alpha \rightarrow V_{i\alpha}$, $i > 1$, such that for any A -algebra A' and for any $v, w \in V_\alpha \otimes_A A'$ one has*

$$(1) \quad X_\alpha(v)X_\alpha(w) = X_\alpha(v+w) \prod_{i \geq 1} X_{i\alpha}(q_\alpha^i(v, w)).$$

(ii) *For any $g \in L_P(A)$, there exist degree i homogeneous polynomial maps $\varphi_{g,\alpha}^i: V_\alpha \rightarrow V_{i\alpha}$, $i \geq 1$, such that for any A -algebra A' and for any $v \in V_\alpha \otimes_A A'$ one has*

$$gX_\alpha(v)g^{-1} = \prod_{i \geq 1} X_{i\alpha}(\varphi_{g,\alpha}^i(v)).$$

If g is contained in the central subtorus $\text{rad}(L)(A)$, then $\varphi_{g,\alpha}^1$ is multiplication by a scalar, and all $\varphi_{g,\alpha}^i$, $i > 1$, are trivial.

(iii) (generalized Chevalley commutator formula) For any $\alpha, \beta \in \Phi_P$ such that $m\alpha \neq -k\beta$ for any $m, k \geq 1$, there exist polynomial maps

$$N_{\alpha\beta ij}: V_\alpha \times V_\beta \rightarrow V_{i\alpha+j\beta}, \quad i, j > 0,$$

homogeneous of degree i in the first variable and of degree j in the second variable, such that for any A -algebra A' and for any $u \in V_\alpha \otimes_A A'$, $v \in V_\beta \otimes_A A'$ one has

$$(2) \quad [X_\alpha(u), X_\beta(v)] = \prod_{i,j>0} X_{i\alpha+j\beta}(N_{\alpha\beta ij}(u, v))$$

If $\text{rank } \Phi_P \geq 2$, for any A -algebra A' we have

$$E(A') = E_P(A') = \langle U_P(A), U_{P^-}(A) \rangle = \langle X_\alpha(V_\alpha \otimes_A A'), \alpha \in \Phi_P \rangle.$$

For any $\alpha \in \Phi_P$, we denote by $U_{(\alpha)}$ the closed subscheme $\prod_{k \geq 1} X_{k\alpha}$ of G so that we have

$$U_{(\alpha)}(A') = \langle X_{k\alpha}(V_{k\alpha} \otimes_A A'), k \geq 1 \rangle$$

for any A'/A . Here $X_{k\alpha}$ is assumed to be trivial if $k\alpha \notin \Phi_P$. This notation coincides with that of [BT] in case of isotropic reductive groups over a field.

Definition 2. Let I be any ideal of the ring A . We denote

$$\begin{aligned} G(A, I) &= \ker(G(A) \rightarrow G(A/I)), \\ E_P^*(A, I) &= G(A, I) \cap E_P(A), \\ E_P(I) &= \langle X_\alpha(IV_\alpha), \alpha \in \Phi_P \rangle = \langle X_\alpha(V_\alpha \otimes_A I), \alpha \in \Phi_P \rangle, \\ E_P(A, I) &= E_P(I)^{E_P(A)} = \text{the normal closure of } E_P(I) \text{ in } E_P(A). \end{aligned}$$

For any $\alpha \in \Phi_P$ there exists by [SGA3, Exp. XXVI Prop. 6.1] a closed connected smooth subgroup G_α of G such that for any $s \in \text{Spec } A$, $(G_\alpha)_{\overline{k(s)}}$ is the standard reductive subgroup of $G_{\overline{k(s)}}$ corresponding to the root subsystem $\pi^{-1}(\{\pm\alpha\} \cup \{0\}) \cap \Phi$. The group G_α is an isotropic reductive group “of isotropic rank 1”, having two opposite parabolic subgroups $L_P \cdot U_{(\alpha)}$ and $L_P \cdot U_{(-\alpha)}$.

We denote by $E_\alpha(A)$ the subgroup of $G(A)$ generated by $U_{(\alpha)}(A)$ and $U_{(-\alpha)}(A)$. Note that we don't know if $E_\alpha(A)$ is normal in $G_\alpha(A)$, and, generally speaking, it depends on the choice of the initial parabolic subgroup P of G .

3. NORMAL SUBGROUPS GENERATED BY ROOT ELEMENTS

In this section we prove Theorem 2.

3.1. Some properties of relative roots. In this section we recall the notion of an (abstract) system of relative roots introduced in [PSt1] and prove some lemmas.

Let Φ be a reduced root system in a Euclidean space with a scalar product $(-, -)$. Let $\Pi = \{a_1, \dots, a_l\}$ be a fixed system of simple roots of Φ ; if Φ is irreducible, we assume that the numbering follows Bourbaki [Bu]. Let D be the Dynkin diagram of Φ . We identify nodes of D with the corresponding simple roots in Π . For a subgroup $\Gamma \subseteq \text{Aut}(D)$ and a Γ -invariant subset $J \subseteq \Pi$, consider the projection

$$\pi = \pi_{J,\Gamma}: \mathbb{Z}\Phi \longrightarrow \mathbb{Z}\Phi / \langle \Pi \setminus J; a - \sigma(a), a \in J, \sigma \in \Gamma \rangle.$$

The set $\Phi_{J,\Gamma} = \pi(\Phi) \setminus \{0\}$ is called the system of *relative roots* corresponding to the pair (J, Γ) . The *rank* of $\Phi_{J,\Gamma}$ is the rank of $\pi(\mathbb{Z}\Phi)$ as a free abelian group.

It is clear that any relative root $\alpha \in \Phi_{J,\Gamma}$ can be represented as a unique linear combination of relative roots from $\pi(\Pi) \setminus \{0\}$. We call the elements of $\pi(\Pi) \setminus \{0\}$ the *simple relative roots*. We say that $\alpha \in \Phi_{J,\Gamma}$ is a *positive* (resp. *negative*) relative root, if it is a non-negative (respectively, a non-positive) linear combination of simple relative roots. The sets of positive and negative relative roots will be denoted by $\Phi_{J,\Gamma}^+$ and $\Phi_{J,\Gamma}^-$ respectively. The *height* $ht(\alpha)$ of a relative root α is the sum of coefficients in its decomposition into a linear combination of simple relative roots (the same thing was called the *level* in [PSt1, LSt]).

Observe that Γ acts on the set of irreducible components of the root system Φ . If this action is transitive, the system of relative roots $\Phi_{J,\Gamma}$ is *irreducible*. Clearly, any system of relative roots $\Phi_{J,\Gamma}$ is a disjoint union of irreducible ones; we call them the *irreducible components* of $\Phi_{J,\Gamma}$.

Let $S \subseteq \Phi$ be any subset. We say that a root $a \in S$ is *maximal* with respect to Π , if there is no simple root $b \in \Pi$ such that $a + b \in S$. In particular, a maximal root of Φ is a root of maximal height in one of the irreducible components of Φ . Roots minimal with respect to Π , and maximal and minimal relative roots in $S \subseteq \Phi_{J,\Gamma}$ are defined similarly. Each irreducible component of $\Phi_{J,\Gamma}$ contains a unique relative root \tilde{a} of maximal height, which is the image under π of a maximal root in Φ .

For any root $a \in \Phi$ we write

$$a = \sum_{b \in \Pi} m_b(a)b, \quad m_b(a) \in \mathbb{Z}, \quad \text{and} \quad a_J = \sum_{b \in J} m_b(a)b.$$

For any two subsets $S_1, S_2 \subseteq \Phi_{J,\Gamma}$ we denote

$$[S_1, S_2] = \{i\alpha + j\beta \mid \alpha \in S_1, \beta \in S_2, i, j > 0\} \cap \Phi_P.$$

Lemma 1. *Let Φ° be an irreducible component of Φ , and $\alpha \in \Phi_{J,\{\text{id}\}}$ be a relative root.*

(i) *For any $\alpha \in \Phi_{J,\{\text{id}\}}$ the set $\Phi^\circ \cap \pi^{-1}(\alpha)$ contains a unique maximal and a unique minimal root.*

(ii) *If $i\alpha, j\alpha \in \Phi_{J,\{\text{id}\}}$ for some $i, j > 0$, then the difference of maximal (respectively, minimal) roots in $\pi^{-1}(i\alpha) \cap \Phi^\circ$ and $\pi^{-1}(j\alpha) \cap \Phi^\circ$ is also a root.*

(iii) *There is an integer $m_\alpha \geq 1$ such that $\mathbb{Z}\alpha \cap \Phi_{J,\Gamma} = \{\pm\alpha, \pm 2\alpha, \dots, \pm m_\alpha\alpha\}$.*

Proof. The proof of (i) and (ii) for the case of maximal roots is literally the same as in [KazS, Lemma 1], which establishes the case where α is a simple relative root. The case of minimal roots is treated similarly. Claim (iii) follows immediately from (ii). \square

Lemma 2. *Let $\alpha \in \Phi_{J,\Gamma}$ be a relative root, and $\beta \in \pi(\Pi) \setminus \{0\}$ a simple relative root such that α and β are linearly independent.*

(i) *There are such integers $m, n \geq 0$ that $\alpha + \mathbb{Z}\beta \cap \Phi_{J,\Gamma} = \{\alpha + i\beta \mid -m \leq i \leq n\}$.*

(ii) *Take $i, j \geq 1$, and let $k, l \in \mathbb{Z}$ be the maximal integers such that $i\alpha + k\beta \in \Phi_{J,\Gamma}$ and $j\alpha + l\beta \in \Phi_{J,\Gamma}$. Then $(j\alpha + l\beta) - (i\alpha + k\beta) \in \Phi_{J,\Gamma}$.*

Proof. Fix $b \in \pi^{-1}(\beta) \cap \Pi$ and $a \in \Phi^+ \cap \pi^{-1}(\alpha)$. Set $\theta = \pi_{J \setminus \Gamma, b, \Gamma}(a) \in \Phi_{J \setminus \Gamma, b, \Gamma}$ and $S = \pi_{J \setminus \Gamma, b, \Gamma}^{-1}(\theta) \cap \Phi$. We can assume that Φ is irreducible and $\alpha \in \Phi_{J,\Gamma}^+$ without loss of generality.

Let $m, n \geq 0$ be the maximal integers such that $\alpha + n\beta$ and $\alpha - m\beta$ are relative roots. By [PSt1, Lemma 3] we have

$$S = \bigcup_{\sigma \in \Gamma} \{x \in \Phi \mid x_{J \setminus \Gamma b} = \sigma(a_{J \setminus \Gamma b})\} = \bigcup_{\sigma \in \Gamma} \sigma(\{x \in \Phi \mid x_{J \setminus \Gamma b} = a_{J \setminus \Gamma b}\}) = \bigcup_{\sigma \in \Gamma} \sigma\left(\pi_{J \setminus \Gamma b, \text{id}}^{-1}(\pi_{J \setminus \Gamma b, \text{id}}(a))\right).$$

By Lemma 1 $\pi_{J \setminus \Gamma b, \text{id}}^{-1}(\pi_{J \setminus \Gamma b, \text{id}}(a))$ contains a unique Π -maximal and a unique Π -minimal root; hence all Π -maximal and Π -minimal roots of S are Γ -conjugate.

On the other hand, note that

$$S = \bigcup_{\substack{i \in \mathbb{Z}: \\ \alpha + i\beta \in \Phi_{J, \Gamma}}} \pi^{-1}(\alpha + i\beta) \cap \Phi.$$

Then, by the choice of m and n , any Π -maximal (respectively, Π -minimal) root of $\pi^{-1}(\alpha + n\beta)$ (respectively, $\pi^{-1}(\alpha - m\beta)$) is a Π -maximal (respectively, Π -minimal) root of S , and by the Γ -conjugacy proved above, all Π -maximal (respectively, Π -minimal) roots of S are of this form.

Let a_0 be any Π -minimal root in S , then there is a Π -maximal root b_0 in S and a sequence of simple roots $c_1, \dots, c_k \in (\Pi \setminus J) \cup \Gamma b$ such that $a_i = a_0 + c_1 + \dots + c_i$ belongs to S for any $1 \leq i \leq k$, and $a_0 + c_1 + \dots + c_k = b_0$. Since $\pi(a_0) = \alpha - m\beta$, $\pi(b_0) = \alpha + n\beta$, it is clear that for each integer $-m \leq i \leq n$ there is an index j , $1 \leq j \leq k$, such that $\pi(a_j) = \alpha + i\beta$. This finishes the proof of (i).

To settle (ii), note that for any roots $a_1 \in \pi^{-1}(i\alpha + k\beta)$ and $a_2 \in \pi^{-1}(j\alpha + l\beta)$, one has $\pi_{J \setminus \Gamma b, \Gamma}(a_1) = i\theta$ and $\pi_{J \setminus \Gamma b, \Gamma}(a_2) = j\theta$. One proves exactly as in part (i) that any Π -maximal root of $\pi_{J \setminus \Gamma b, \Gamma}^{-1}(i\theta)$ belongs to $\pi^{-1}(i\alpha + k\beta)$, and any Π -maximal root of $\pi_{J \setminus \Gamma b, \Gamma}^{-1}(j\theta)$ belongs to $\pi^{-1}(j\alpha + l\beta)$. On the other hand, by Lemma 1 there is a pair of maximal roots in $\pi_{J \setminus \Gamma b, \Gamma}^{-1}(j\theta)$ and $\pi_{J \setminus \Gamma b, \Gamma}^{-1}(i\theta)$ whose difference is a root. This implies that $(j\alpha + l\beta) - (i\alpha + k\beta) \in \Phi_{J, \Gamma}$. \square

Lemma 3. *Let $\tilde{\alpha} \in \Phi_{J, \Gamma}$ be a maximal with respect to $\pi(\Pi) \setminus \{0\}$ relative root. Let $\gamma \in \pi(\Pi) \setminus \{0\}$ be a simple relative root. If $i\tilde{\alpha} + j\gamma \in \Phi_{J, \Gamma}$ for some $i, j \in \mathbb{Z}$, then $i \in \{0, \pm 1\}$, and if $i \neq 0$, then i and j have opposite sign.*

Proof. Let $\delta \in \pi(\Pi) \setminus \{0\}$ be a simple relative root different from γ in the same irreducible component of $\Phi_{J, \Gamma}$ as $\tilde{\alpha}$. Assume that the coefficient of δ in the decomposition of $\tilde{\alpha}$ is n . Since $\tilde{\alpha}$ is a maximal positive relative root, $n > 0$.

If $i > 1$, the coefficient of δ in the decomposition of $i\tilde{\alpha} + j\gamma$ is $in > n$, which is clearly impossible, since any absolute root in $\pi^{-1}(i\tilde{\alpha} + j\gamma)$ is dominated by a maximal root of Φ , belonging to $\pi^{-1}(\tilde{\alpha})$. The case $i < -1$ is symmetric.

If $i = \pm 1$, then i and j have opposite sign, since $\tilde{\alpha}$ is a maximal root. \square

Lemma 4. *Assume that $\text{rank}(\Phi_{J, \Gamma}) \geq 2$ and it is irreducible. Let $\tilde{\alpha} \in \Phi_{J, \Gamma}$ be the root of maximal height with respect to $\pi(\Pi) \setminus \{0\}$. For any relative root $\alpha_0 \in \Phi_{J, \Gamma}$ there are relative roots $\alpha_1, \dots, \alpha_n \in \Phi_{J, \Gamma}^+$ satisfying the following conditions:*

- the sum $\alpha_0 + \alpha_1 + \dots + \alpha_i$ is a relative root for any $1 \leq i \leq n$;
- $\alpha_0 + \alpha_1 + \dots + \alpha_n = \tilde{\alpha}$;

- for any $0 \leq i \leq n$ there are no positive integers m, n such that $m(\alpha_0 + \alpha_1 + \dots + \alpha_i) = -n\alpha_{i+1}$.

Moreover, if $\alpha_0 \in \Phi_{J,\Gamma}^+$, then all roots α_i , $1 \leq i \leq n$, can be chosen in $\pi(\Pi) \setminus 0$. If $\alpha_0 \in \Phi_{J,\Gamma}^-$, then α_1 can be chosen so that α_0 and α_1 are linearly independent.

Proof. Pick a root $a \in \pi^{-1}(\alpha_0)$, and let \tilde{a} be the root of maximal height in Φ with respect to Π , lying in the same irreducible component of Φ as a . Clearly, $\pi(\tilde{a}) = \tilde{\alpha}$. We prove the claim by induction on $d(a) = ht(\tilde{a}) - ht(a)$. If $d(a) = 0$, there is nothing to prove. Assume that $d(a) > 0$.

Assume first that $\alpha \in \Phi_{J,\Gamma}^+$ and $a \in \Phi^+$. There is a sequence of simple roots $c_1, \dots, c_l \in \Pi$ such that each sum $a + c_1 + \dots + c_i$ is a root and $a + c_1 + \dots + c_l = \tilde{a}$. Let $1 \leq i_1 < i_2 < \dots < i_n \leq l$ be all indices such that $\pi(c_{i_j}) \neq 0$. Set $\beta_j = \pi(a + c_1 + \dots + c_{i_j})$ for all j and $\alpha_j = \pi(c_{i_j})$. Since all roots involved are positive, β_j and α_{j+1} are not opposite-directional. On the other hand,

$$\beta_j + \alpha_{j+1} = \pi(a + c_1 + \dots + c_{i_j} + c_{i_{j+1}}) = \pi(a + c_1 + \dots + c_{i_{j+1}})$$

is a relative root. Thus we get a sequence of relative roots $\beta_j = \alpha_0 + \alpha_1 + \dots + \alpha_j$, $1 \leq j \leq n$, such that $\tilde{\alpha} = \beta_n$. Clearly, these roots satisfy the conditions of the Lemma.

Assume now that $a \in \Phi^-$ and $\alpha_0 = \pi(a) = -n\pi(b)$ for a simple root $b \in J$ and a positive integer n . Denote by S the set of simple roots in Π that occur in the decomposition of a with nonzero coefficients. Since $\text{rank}(\Phi_{J,\Gamma}) \geq 2$, we have $J \setminus \Gamma b \neq \emptyset$. Clearly, $(J \setminus \Gamma b) \cap S = \emptyset$. Let c be the sum of simple roots in $\Pi \setminus S$, that form a chain on the Dynkin digram of Φ starting with a simple root in $J \setminus \Gamma b$ and ending next to an element of S . Clearly, c is a positive root and $(c, -a) < 0$, hence $c - a$ is also a positive root. Apart from that, since c contains in its decomposition a root of $J \setminus \Gamma b$, we have $\pi(c) \neq 0$, $\pi(c - a) \neq 0$, and $\pi(c)$ is non-collinear to $\pi(c - a)$. Then $\alpha_1 = \pi(c - a)$ satisfies all requirements of the lemma. We have $c \in \pi^{-1}(\alpha_0 + \alpha_1)$, and $d(c) = d(a) - ht(c - a) < d(a)$, hence we can apply to c the induction hypothesis. This gives a chain of relative roots $\gamma_1, \dots, \gamma_m \in \Phi_{J,\Gamma}^+$ such that $(\alpha_0 + \alpha_1) + \gamma_1 + \dots + \gamma_i$ is a relative root for any $1 \leq i \leq n$, $(\alpha_0 + \alpha_1) + \gamma_1 + \dots + \gamma_m = \tilde{\alpha}$, and $(\alpha_0 + \alpha_1) + \gamma_1 + \dots + \gamma_i$ and γ_{i+1} are not opposite-directional. Setting $\alpha_i = \gamma_{i-1}$, we obtain the desired sequence of relative roots.

Finally, assume that $a \in \Phi^-$ and $\alpha_0 = \pi(a) \neq -m\pi(b)$ for any $b \in J$. There is a simple root $b \in \Pi$ such that $a + b \in \Phi^-$; clearly, $d(a + b) < d(a)$. If $b \in \Pi \setminus J$, then $\pi(a + b) = \alpha_0$ and we just substitute a by $a + b$ and apply the induction hypothesis. If $b \in J$, then we set $\alpha_1 = \pi(b)$. Clearly, α_0 and α_1 are linearly independent. We apply the induction hypothesis to $a + c$ to obtain the roots $\alpha_2, \dots, \alpha_n$ with required properties. \square

Definition 3. Let $\delta, \gamma \in \Phi_{J,\Gamma}$ be two relative roots. We call a sequence of positive relative roots $\beta_1, \dots, \beta_n \in \Phi_{J,\Gamma}^+$ a special chain between δ and γ , if it satisfies the following properties:

- $\delta + \beta_1 + \dots + \beta_n = \gamma$;
- $\delta + \beta_1 + \dots + \beta_i \in \Phi_{J,\Gamma}$ for any $1 \leq i \leq n$;
- for any $0 \leq i \leq n$ one has $0 \notin [[\dots [\delta, \beta_1], \dots, \beta_{i-1}], \beta_i]$.
- for any $0 \leq i \leq n$ the relative root $\delta + \beta_1 + \dots + \beta_{i+1}$ is $\pi(\Pi) \setminus \{0\}$ -minimal in the set of relative roots of the form $k(\delta + \beta_1 + \dots + \beta_i) + l\beta_{i+1}$, $k, l > 0$.

We call a sequence of negative relative roots $\beta_1, \dots, \beta_n \in \Phi_{J,\Gamma}^-$ a special chain between δ and γ , if $-\beta_1, \dots, -\beta_n$ is a special chain between $-\delta$ and $-\gamma$ in the above sense.

Lemma 5. Assume that $\text{rank}(\Phi_{J,\Gamma}) \geq 2$ and it is irreducible. Let $\tilde{a} \in \Phi$ be a maximal root, and set $\tilde{\alpha} = \pi(\tilde{a})$.

There is a positive root $s \in \Phi$, such that $\sigma = \pi(s)$ is a simple relative root, $\tilde{a} - s \in \Phi^+$, and either (a) or (b) below takes place.

(a) $\tilde{\alpha} - k\sigma \notin \Phi_{J,\Gamma}$ for any $k \geq 2$; then $(\mathbb{Z}\tilde{\alpha} + \mathbb{Z}\sigma) \cap \Phi_{J,\Gamma} = \{\pm\tilde{\alpha}, \pm\sigma, \pm(\tilde{\alpha} - \sigma)\}$, and, in particular, $[\tilde{\alpha}, -\sigma] = \{\tilde{\alpha} - \sigma\}$; $[\tilde{\alpha} - \sigma, -\tilde{\alpha}] = \{-\sigma\}$, and $[-\sigma, -\tilde{\alpha} + \sigma] = \{-\tilde{\alpha}\}$.

(b) the maximal integer k such that $\tilde{\alpha} - k\sigma \in \Phi_{J,\Gamma}$ is ≥ 2 ; one has $(k-1)\sigma \in \Phi_{J,\Gamma}$, $i\sigma \notin \Phi_{J,\Gamma}$ for any $i \geq k+1$, and

$$(3) \quad \begin{aligned} [\tilde{\alpha}, -\sigma] &= \{\tilde{\alpha} - \sigma, \tilde{\alpha} - 2\sigma, \dots, \tilde{\alpha} - k\sigma\}, & [[\tilde{\alpha}, -\sigma], -(k-1)\sigma] &= \{\tilde{a} - k\sigma\}, \\ [\tilde{\alpha} - k\sigma, -\tilde{\alpha} + \sigma] &= \{-(k-1)\sigma\} \cup \{-\tilde{\alpha} \mid k=2\}, \\ [[\tilde{\alpha} - k\sigma, -\tilde{\alpha} + \sigma], -\tilde{\alpha} + (k-1)\sigma] &= \{-\tilde{\alpha}\}. \end{aligned}$$

Moreover, let $(\alpha_1, \dots, \alpha_n)$ be the sequence of relative roots $(-\sigma, -\tilde{\alpha}, -\tilde{\alpha} + \sigma)$ or, respectively, $(-\sigma, -(k-1)\sigma, -\tilde{\alpha} + \sigma, -\tilde{\alpha} + (k-1)\sigma)$, depending on whether (a) or (b) takes place. Then this sequence is a special chain between $\tilde{\alpha}$ and $-\tilde{\alpha}$, and there are roots $a_i \in \pi^{-1}(\alpha_i)$, $1 \leq i \leq n$, such that $\tilde{a} + a_1 + \dots + a_i$ is a root for any $1 \leq i \leq n$, and $\tilde{a} + a_1 + \dots + a_n = -\tilde{a}$.

Proof. We can assume from the start that Φ is irreducible. Let D be the Dynkin diagram of Φ and \tilde{D} be the extended Dynkin diagram. Let s be the sum of simple roots in Π forming a chain in \tilde{D} that starts with the root from J nearest to $-\tilde{a}$, and ends next to $-\tilde{a}$; denote the corresponding root from J by s_0 . Clearly, $(\tilde{a}, -s) > 0$, hence $\tilde{a} - s \in \Phi$. Since $|J| \geq 2$, not all simple roots of Φ are involved in s , and therefore $\tilde{a} - s \in \Phi^+$. Since $\pi(s) = \pi(s_0)$, it is a simple relative root σ . Since $\pi(\tilde{a} - s) = \tilde{\alpha} - \sigma$, we have $\tilde{\alpha} - \sigma \in \Phi_{J,\Gamma}^+$.

Case (a). If $\tilde{\alpha} - k\sigma \notin \Phi_{J,\Gamma}$ for any $k \geq 2$, then $(\mathbb{Z}\tilde{\alpha} + \mathbb{Z}\sigma) \cap \Phi_{J,\Gamma} = \{\pm\tilde{\alpha}, \pm\sigma, \pm(\tilde{\alpha} - \sigma)\}$ by Lemma 3. Let c be a Π -minimal root in $\pi^{-1}(-\sigma)$. If $S \subseteq D \setminus \{J \setminus s_0\}$ is the connected component containing s_0 , then c is a negative linear combination of some simple roots in S . Since $-\tilde{a}$ is adjacent to a root of S in \tilde{D} , we have $(-\tilde{a}, c) > 0$, or, equivalently, $(\tilde{a}, c) < 0$. Hence $\tilde{a} + c \in \pi^{-1}(\tilde{\alpha} - \sigma) = \pi^{-1}(\tilde{\alpha} + \alpha_1)$ is a root, choose $a_1 = c$ and $a_2 = -\tilde{a}$, so that $\tilde{a} + a_1 + a_2 = c$, and then $a_3 = -\tilde{a} - c$.

Case (b). Now assume that the maximal k such that $\tilde{\alpha} - k\sigma \in \Phi_{J,\Gamma}$ is ≥ 2 . We claim that $\tilde{\alpha} - i\sigma \in \Phi_{J,\Gamma}$ for all $1 \leq i \leq k-1$. Indeed, by Lemma 4 there is a chain of simple relative roots γ_j , $1 \leq j \leq n$, such that $\beta_j = (\tilde{\alpha} - k\sigma) + \gamma_1 + \dots + \gamma_j \in \Phi_{J,\Gamma}$ for all j , and $(\tilde{\alpha} - k\sigma) + \gamma_1 + \dots + \gamma_n = \tilde{\alpha}$. Since σ is itself a simple relative root, we necessarily have $\gamma_1 = \dots = \gamma_n = \sigma$.

Let $S \subseteq \tilde{D} \setminus \{J \setminus s_0\}$ be the connected component containing $-\tilde{a}$ and s_0 , and let Δ be the root subsystem of Φ generated by S . Let $c \in \pi^{-1}(\tilde{\alpha} - k\sigma)$ be the root of maximal height in Δ with respect to $-S$. Choose $b \in \Phi^+$ to be the sum of all roots in $-S$, then $\pi(b) = \pi(\tilde{a} - s_0) = \tilde{\alpha} - \sigma$. Considering the extended Dynkin diagram of Δ , we conclude that $(c, b) > 0$, and hence $c - b \in \Phi$. Consequently, $(k-1)\sigma \in \Phi_{J,\Gamma}$.

We have $i\sigma \notin \Phi_{J,\Gamma}$ for any $i \geq k+1$, since if it is the case, $\tilde{a} - i\sigma$ is a relative root. The equalities (3) now follow from Lemma 3.

Note that we have $b - \tilde{a} \in \Phi$ as well. Indeed, the root system Φ is not of type A_l , because it is only possible in case (a). Then $-\tilde{a}$ is a leaf of \tilde{D} and S . Then $b - \tilde{a}$ is minus the sum of all simple roots in S except $-\tilde{a}$, which is a root.

Moreover, $c - (b - \tilde{a})$ is also a root. Indeed, since $c - (b - \tilde{a}) - \tilde{a} = c - b$ is a root, by [PSt1, Lemma 1] we have that if $c - (b - \tilde{a}) \notin \Phi$, then necessarily $c - \tilde{a} \in \Phi$. Both \tilde{a} and c are long roots in Φ , hence if $c - \tilde{a} \in \Phi$, we have $(\tilde{a}, c) = -1/2$, assuming both \tilde{a} and c have length 1. However, $(\tilde{a}, c) = (c - x, c) = 1 - (x, c)$, where x is a positive linear combination of roots in $-S \setminus \{\tilde{a}\}$. Therefore, $(c, d) > 0$ for at least one root d from $-S \setminus \{\tilde{a}\}$, which implies $(c, b - \tilde{a}) < 0$ and $c - (b - \tilde{a}) \in \Phi$.

We set $a_1 = b - \tilde{a} \in \pi^{-1}(-\sigma)$, $a_2 = c - b \in \pi^{-1}(-(k-1)\sigma)$, $a_3 = -b \in \pi^{-1}(-\tilde{\alpha} + \sigma)$, $a_4 = -\tilde{a} + b - c \in \pi^{-1}(-\tilde{\alpha} + (k-1)\sigma)$.

It is easy to see in both cases that the sequence $\alpha_1, \dots, \alpha_n$ is a special chain. □

Lemma 6. *Assume that $\text{rank}(\Phi_{J,\Gamma}) \geq 2$ and it is irreducible. Let $\tilde{a} \in \Phi$ be a maximal root with respect to Π , and let $a_1, \dots, a_n \in \Phi^+$ be a sequence of roots such that the sum $\tilde{a} - a_1 - \dots - a_i$ is a root and $\pi(a_i) \neq 0$ for all $1 \leq i \leq n$, and $b = \tilde{a} - a_1 - \dots - a_n$ is a Π -minimal root in $\pi^{-1}(\pi(b))$, where $\pi(b) \neq 0$. For any $a_0 \in \Phi$ satisfying $\pi(a_0) = \pi(\tilde{a})$ and belonging to the same irreducible component of Φ as \tilde{a} , there is a sequence of positive roots $a'_i \in \pi^{-1}(\pi(a_i))$, $1 \leq i \leq n$, such that $a_0 - a'_1 - \dots - a'_n = b$ and any sum $a_0 - a'_1 - \dots - a'_i$ is a root.*

Proof. For any root $a_0 \in \Phi$ belonging to the same irreducible component as \tilde{a} , there is a sequence of simple roots $b_1, \dots, b_k \in \Pi$ such that each sum $a_0 + b_1 + \dots + b_i$ is a root and $a_0 + b_1 + \dots + b_k = \tilde{a}$. If $a_0 \in \pi^{-1}(\pi(\tilde{a}))$, moreover, all $b_i \in \Pi \setminus J$.

The equality $\tilde{a} = b + a_1 + a_2 + \dots + a_n$ is rewritten as

$$(4) \quad a_0 + b_1 + \dots + b_k = b + a_1 + a_2 + \dots + a_n.$$

We prove that this last equality implies the equality

$$a_0 + b_1 + \dots + b_{k-1} = b + a'_1 + a'_2 + \dots + a'_n$$

for some new positive roots a'_i such that $\pi(a'_i) = \pi(a_i)$, $1 \leq i \leq n$. The claim of the lemma then follows by induction on k .

The equality (4) together with the definition of b_i 's implies that $(b + a_1 + \dots + a_{n-1}) + a_n - b_k$ is a root. Observe that none of the three roots $b + a_1 + \dots + a_{n-1}$, a_n and $-b_k$ is opposite to another. Indeed, we know that $(b + a_1 + \dots + a_{n-1}) + a_n = \tilde{a} \neq 0$. Since $b_k \in \Pi \setminus J$ and $\pi(a_n) \neq 0$, a_n and b_k are linearly independent. Since all a_i are positive and $\pi(b) \neq 0$, we have $\pi(b + a_1 + \dots + a_{n-1}) \neq 0$ as well, hence $b + a_1 + \dots + a_{n-1}$ and b_k are linearly independent too. Then we can apply [PSt1, Lemma 1], which tells that at least one of the expressions $(b + a_1 + \dots + a_{n-1}) - b_k$ and $a_n - b_k$ is a root too. In the first case, applying the same lemma several more times, we eventually conclude that $a_i - b_k$ is a root for some $1 \leq i \leq n-1$, since $b - b_k$ is not a root for any positive simple root b_k by the minimality of b . Summing up, we see that $a_i - b_k$ is a root for some $1 \leq i \leq n$. Therefore, $a_0 + b_1 + \dots + b_{k-1} = b + a'_1 + \dots + a'_n$, where $a'_i = a_i - b_k \in \pi^{-1}(\pi(a_i))$ and $a'_j = a_j$ for all $j \neq i$. Note that $b + a'_1 + \dots + a'_l$ is still a root for any $1 \leq l \leq n$ by the choice of the index i . □

Lemma 7. *Assume that $\text{rank}(\Phi_{J,\Gamma}) \geq 2$ and it is irreducible. Let $\tilde{\alpha} \in \Phi_{J,\Gamma}$ be the maximal root. Then there are special chains between $-\tilde{\alpha}$ and $\tilde{\alpha}$, and between α and $\tilde{\alpha}$ for any $\alpha \in \Phi_{J,\Gamma}^+$.*

Proof. The existence of a special chain between $-\tilde{\alpha}$ and $\tilde{\alpha}$ follows from Lemma 5. The existence of a special chain between α and $\tilde{\alpha}$ for any $\alpha \in \Phi_{J,\Gamma}^+$ follows from Lemma 4. \square

3.2. Proof of Theorem 2. We assume that G is an isotropic reductive group over a commutative ring R , P is a parabolic subgroup of G . For simplicity, we assume that the types of G and P are constant over R .

Let Φ_P be the system of relative roots for P , and $X_\alpha(V_\alpha)$, $\alpha \in \Phi_P$, the relative root subschemes of G with respect to P and an opposite parabolic subgroup P^- ; set $L = P^+ \cap P^-$ be their common Levi subgroup.

Fix a system of simple roots $\Pi \subseteq \Phi$ such that P is standard with respect to it. Let $J \subseteq \Pi$ be the subset corresponding to the type of P , and Γ be the subgroup of the automorphism group of the Dynkin diagram of Φ representing the $*$ -action. We identify $\Phi_P = \Phi_{J,\Gamma}$ and denote by

$$\pi_P = \pi_{J,\Gamma} : \Phi \rightarrow \Phi_P \cup \{0\}$$

the canonical surjective map.

We assume from now on that $\text{rank } \Phi_P \geq 2$.

Fix a subgroup H of $G(R)$ normalized by $E(R)$. For any $\alpha \in \Phi_P$ we define subsets $M_\alpha \subseteq V_\alpha$ by the equality

$$H \cap X_\alpha(V_\alpha) = X_\alpha(M_\alpha).$$

Lemma 8. *Let G be a reductive group over a commutative ring R , P a parabolic subgroup of G , $A, B \in \Phi_P$ two non-proportional relative roots such that $A + B \in \Phi_P$. Assume that the structure constants of Φ are invertible in R , or that $A - B \notin \Phi_P$. Take $0 \neq u \in V_B$. Any generating system e_1, \dots, e_n of the R -module V_A contains an element e_i such that $N_{AB11}(e_i, u) \neq 0$.*

Proof. Assume that $N_{AB11}(e_i, u) = 0$ for all $1 \leq i \leq n$. Consider an affine fpqc-covering $\coprod \text{Spec } S_\tau \rightarrow \text{Spec } R$ that splits G . There is a member $S_\tau = S$ of this covering such that the image of $X_B(u)$ under $G(R) \rightarrow G(S)$ is non-trivial. Write

$$X_B(u) = \prod_{\pi(\beta)=B} x_\beta(a_\beta) \cdot \prod_{i \geq 2} \prod_{\pi(\beta)=iB} x_\beta(c_\beta),$$

where $\pi : \Phi \rightarrow \Phi_P$ is the canonical projection of the absolute root system of G onto the relative one, x_β are root subgroups of the split group G_S , and $a_\beta \in S$. Since $X_B(u) \neq 0$, the definition of X_B implies that there exists $a_\beta \neq 0$. Let $\beta_0 \in \pi^{-1}(B)$ be the root of minimal height with this property. By [PSt1, Lemma 4] there exists a root $\alpha \in \pi^{-1}(A)$ such that $\alpha + \beta_0 \in \Phi$. Let $v \in V_A \otimes_R S$ be such that $X_A(v) = x_\alpha(1) \prod_{i \geq 2} \prod_{\pi(\gamma)=iA} x_\gamma(d_\gamma)$, for some $d_\gamma \in S$. Then the (usual) Chevalley com-

mutator formula implies that $[X_A(v), X_B(u)]$ contains in its decomposition a factor $x_{\alpha+\beta}(\lambda a_{\beta_0})$, where $\lambda \in \{\pm 1, \pm 2, \pm 3\}$. However, since either the structure constants of Φ are invertible, or $A - B \notin \Phi_P$, we have $\lambda \in R^\times$. Then $N_{AB11}(v, u) \neq 0$, a contradiction. \square

Definition 4. Let $\alpha_1, \dots, \alpha_n \in \Phi_P$ be a sequence of relative roots such that $\alpha_1 + \dots + \alpha_i \in \Phi_P$ for any $1 \leq i \leq n$. We define the n -linear map

$$N_{\alpha_1, \dots, \alpha_n} : V_{\alpha_1} \times \dots \times V_{\alpha_n} \rightarrow V_{\alpha_1 + \dots + \alpha_n}$$

by the recursive formula

$$\begin{aligned} N_{\alpha_1, \alpha_2} &= N_{\alpha_1, \alpha_2, 1, 1}(-, -), \\ N_{\alpha_1, \dots, \alpha_{i+1}}(v_1, \dots, v_n) &= N_{\alpha_1 + \dots + \alpha_i, \alpha_{i+1}, 1, 1}(N_{\alpha_1, \dots, \alpha_i}(v_1, \dots, v_i), v_{i+1}). \end{aligned}$$

Clearly, $N_{\alpha_1, \dots, \alpha_n}$ induces a linear map

$$N'_{\alpha_1, \dots, \alpha_n} : V_{\alpha_1} \otimes \dots \otimes V_{\alpha_n} \rightarrow V_{\alpha_1 + \dots + \alpha_n}.$$

Lemma 9. Let $\alpha_1, \dots, \alpha_n \in \Phi_P^+$ be a special chain between δ and γ . Then

$$[[\dots [X_\delta(u), X_{\alpha_1}(v_1)], \dots], X_{\alpha_n}(v_n)] \in X_\gamma(N_{\delta, \alpha_1, \dots, \alpha_n}(u, v_1, \dots, v_n)) \cdot \prod_{i=1}^k X_{\beta_i}(V_{\beta_i})$$

for some $k \geq 0$ and $\beta_i \in \Phi_P$ such that $ht(\beta_i) > ht(\gamma)$ for any $1 \leq i \leq k$.

Proof. The formula follows by induction on n from the generalized Chevalley commutator formula. \square

Lemma 10. Assume that the structure constants of Φ are invertible in R and $\text{rank } \Phi_P \geq 2$.

Let $\tilde{\alpha}$ be the root of maximal height in Φ_P . Let I be an ideal in R such that $M_{\tilde{\alpha}} \subseteq IV_{\tilde{\alpha}}$. Then for any $\beta \in \Phi_P$ one has $M_\beta \subseteq IV_\beta$.

Proof. After passing from the ring R to R/I , we can assume that $I = \{0\}$, and we need to show that $M_\beta = 0$. Assume that $v_0 \neq 0$ is an element of M_β .

Assume first that there is a special chain $\beta_1, \dots, \beta_k \in \Phi_P^+$ between β and $\tilde{\alpha}$. Set $\beta_0 = \beta$, and $\gamma_l = \sum_{i=0}^l \beta_i$ for all $1 \leq l \leq k$; in particular, $\gamma_k = \tilde{\alpha}$. By Lemma 8 there are non-zero elements $v_i \in V_{\gamma_i}$, $0 \leq i \leq k$, and $u_i \in V_{\beta_i}$, $1 \leq i \leq k$, satisfying the equality

$$v_{i+1} = N_{\gamma_i, \beta_{i+1}, 1, 1}(v_i, u_{i+1}) \quad \text{for all } 0 \leq i \leq k-1.$$

Applying Lemma 9, we conclude that since $v_0 \in M_\beta$, we have $v_k \in M_{\tilde{\alpha}}$, which contradicts $M_{\tilde{\alpha}} = 0$.

Thus, Lemma 7 implies that $M_\beta \subseteq IV_\beta$ for any $\beta \in \Phi_P^+$, and $M_{-\tilde{\alpha}} \subseteq IV_{-\tilde{\alpha}}$. Changing the system of simple roots Π to $-\Pi$ in Φ , we deduce that $M_{-\tilde{\alpha}} \subseteq IV_{-\tilde{\alpha}}$ implies $M_\beta \subseteq IV_\beta$ for all $\beta \in \Phi_P^-$. \square

Lemma 11. Assume that the structure constants of Φ are invertible in R and $\text{rank } \Phi_P \geq 2$. Let $\tilde{\alpha}$ be the root of maximal height in Φ_P . Then $M_{\tilde{\alpha}} = IV_{\tilde{\alpha}}$ for some ideal $I \subseteq R$.

Proof. Since $\tilde{\alpha}$ does not have positive multiples in Φ_P , the set $M_{\tilde{\alpha}}$ is an additive subgroup of $V_{\tilde{\alpha}}$.

Let $\tilde{a} \in \pi^{-1}(\tilde{\alpha})$ be the root of maximal height in Φ . Let $\alpha_1, \dots, \alpha_n \in \Phi_P^-$ be the special chain from \tilde{a} to $-\tilde{a}$ constructed in Lemma 5. By Lemma 9 we have

$$[[\dots [X_{\tilde{a}}(u), X_{\alpha_1}(v_1)], \dots], X_{\alpha_n}(v_n)] = X_{-\tilde{a}}(N_{\tilde{a}, \alpha_1, \dots, \alpha_n}(u, v_1, \dots, v_n))$$

for any $u \in V_{\tilde{\alpha}}$, $v_i \in V_{\alpha_i}$. Applying the same lemma to the special chain $-\alpha_1, \dots, -\alpha_n$ between $-\tilde{\alpha}$ and $\tilde{\alpha}$, we see that

$$N_{\tilde{\alpha}, \alpha_1, \dots, \alpha_n, -\alpha_1, \dots, -\alpha_n} \left(M_{\tilde{\alpha}}, V_{\alpha_1}, \dots, V_{\alpha_n}, V_{-\alpha_1}, \dots, V_{-\alpha_n} \right) \subseteq M_{\tilde{\alpha}}.$$

This implies, in particular, that $M_{\tilde{\alpha}}$ is an R -submodule of $V_{\tilde{\alpha}}$.

Consider the natural R -linear map

$$F : V_{\alpha_1} \otimes \dots \otimes V_{\alpha_n} \otimes V_{-\alpha_1} \otimes \dots \otimes V_{-\alpha_n} \rightarrow \text{End}_R(V_{\tilde{\alpha}})$$

induced by $N_{\tilde{\alpha}, \alpha_1, \dots, \alpha_n, -\alpha_1, \dots, -\alpha_n}$. We are going to show that F is surjective. In order to do that, by faithfully flat descent we can assume that G is split and all root modules V_{β} , $\beta \in \Phi_P$, are free.

By Lemma 5 combined with Lemma 6, for any $a \in \pi^{-1}(\tilde{\alpha})$ there is a sequence of roots $a_i \in \pi^{-1}(\alpha_i)$ such that all sums $a + a_1 + \dots + a_i$ are roots and $a + a_1 + \dots + a_n = -\tilde{\alpha}$. By [PSt1, Theorem 2] there are $v(a) \in V_{\tilde{\alpha}}$, $v_i(a) \in V_{\alpha_i}$, and $u_i(a) \in V_{-\alpha_i}$ such that

$$\begin{aligned} X_{\tilde{\alpha}}(v(a)) &= x_a(1); \quad X_{\alpha_i}(v_i(a)) \in x_{a_i}(1) \prod_{j \geq 2} X_{j\alpha_i}(V_{j\alpha_i}), \quad 1 \leq i \leq n; \\ X_{-\alpha_i}(u_i(a)) &\in x_{-a_i}(1) \prod_{j \geq 2} X_{-j\alpha_i}(V_{-j\alpha_i}), \quad 1 \leq i \leq n. \end{aligned}$$

Using the generalized Chevalley commutator formula and the invertibility of constants, one readily sees that for any $a, a' \in \pi^{-1}(\tilde{\alpha})$

$$X_{\tilde{\alpha}} \left(N_{\tilde{\alpha}, \alpha_1, \dots, \alpha_n, -\alpha_1, \dots, -\alpha_n} (v(a), v_1(a), \dots, v_n(a), u_1(a'), \dots, u_n(a')) \right) = X_{\tilde{\alpha}}(n(a, a')v(a')),$$

where $n(a, a') \in R^\times$.

Note that for any $b \in \pi^{-1}(\tilde{\alpha})$ such that $ht(b) \leq ht(a)$, we have $ht(b + a_1 + \dots + a_n) \leq ht(-\tilde{\alpha})$, which implies that either $b = a$, or $b + a_1 + \dots + a_n$ is not a root. Hence for any $b \neq a$, if $ht(b) \leq ht(a)$, we have

$$N_{\tilde{\alpha}, \alpha_1, \dots, \alpha_n, -\alpha_1, \dots, -\alpha_n} (v(b), v_1(a), \dots, v_n(a), u_1(a'), \dots, u_n(a')) = 0.$$

On the other hand, for any $b \in \pi^{-1}(\tilde{\alpha})$ such that $ht(b) > ht(a)$, if $b + a_1 + \dots + a_n - a'_1 - \dots - a'_n = b - a + a'$ is a root, then

$$N_{\tilde{\alpha}, \alpha_1, \dots, \alpha_n, -\alpha_1, \dots, -\alpha_n} \left(v(b), v_1(a), \dots, v_n(a), u_1(a'), \dots, u_n(a') \right) = m(b, a, a')v(x_{b-a+a'})$$

for some $m(b, a, a') \in R^\times$, and, clearly, $ht(b - a + a') > ht(a')$.

Consider $v(a)$, $a \in \pi^{-1}(\tilde{\alpha})$, as a basis e_1, \dots, e_k , $k = |\pi_P^{-1}(\tilde{\alpha})|$, of $V_{\tilde{\alpha}}$ with a height-respecting numbering. Observations in the previous paragraph imply that the matrix of $F(v_1(a), \dots, v_n(a), u_1(a'), \dots, u_n(a'))$ in this basis has an invertible entry in the position (i_0, j_0) , where $e_{i_0} = v(a')$ and $e_{j_0} = v(a)$, and all other non-zero entries of this matrix are in positions (i, j) with $j > j_0$ and $i > i_0$. This readily implies that the R -module $F(V_{\alpha_1}, \dots, V_{\alpha_n}, V_{-\alpha_1}, \dots, V_{-\alpha_n})$ contains all elementary endomorphisms of $V_{\tilde{\alpha}}$ that send $v(a)$ to $v(a')$ and all other elements of the basis to 0. Hence $F(V_{\alpha_1}, \dots, V_{\alpha_n}, V_{-\alpha_1}, \dots, V_{-\alpha_n}) = \text{End}_R(V_{\tilde{\alpha}})$.

It remains to observe that, since $V_{\tilde{\alpha}}$ is a faithfully (since the type of P is constant) projective R -module, and $M_{\tilde{\alpha}}$ is invariant under $\text{End}_R(V_{\tilde{\alpha}})$, then there is an ideal I of R such that $IV_{\tilde{\alpha}} = M_{\tilde{\alpha}}$.

□

Proof of Theorem 2. By Lemma 11 there are two ideals I, J of R such that $M_{\tilde{\alpha}} = IV_{\tilde{\alpha}}$ and $M_{-\tilde{\alpha}} = JV_{-\tilde{\alpha}}$. By Lemma 10 we conclude that $IV_{\tilde{\alpha}} \subseteq JV_{\tilde{\alpha}}$ and $JV_{-\tilde{\alpha}} \subseteq IV_{-\tilde{\alpha}}$. Since $V_{\pm\tilde{\alpha}}$ is a finitely generated projective module, this implies that $I = J$. Also by Lemma 10, we know that $M_{\alpha} \subseteq IV_{\alpha}$ for any $\alpha \in \Phi_P$.

Now we prove the following claim for any $\alpha \in \Phi_P^+$ by inverse induction on $ht(\alpha)$: for any simple relative root $\beta \in \pi(\Pi) \setminus \{0\}$ linearly independent with α , and any relative root of the form $\alpha - j\beta$, $j \geq 0$, we have $M_{\alpha-j\beta} = IV_{\alpha-j\beta}$.

Take any $\alpha \in \Phi_P^+$. First we note that $M_{\alpha} = IV_{\alpha}$. Indeed, if $\alpha = \tilde{\alpha}$, this is known; otherwise by Lemma 4 there is a simple relative root γ , such that γ is linearly independent with α and $\alpha + \gamma \in \Phi_P^+$. Since $ht(\alpha + \gamma) > ht(\alpha)$, the induction hypothesis indeed implies $M_{\alpha} = M_{(\alpha+\gamma)-\gamma} = IV_{(\alpha+\gamma)-\gamma}$.

By Lemma 2 (i) we have $(\alpha + \mathbb{Z}\beta) \cap \Phi_P = \{\alpha - j\beta \mid -m \leq j \leq n\}$ for some $m, n \geq 0$. Clearly, we can assume that $\alpha + \beta \notin \Phi_P$, i.e. $m = 0$; otherwise we are again reduced to the induction hypothesis. We show that $M_{\alpha-j\beta} = IV_{\alpha-j\beta}$ for any $0 \leq j \leq n$ by the inverse induction on j (the second induction).

Consider the commutator expression

$$[\dots [X_{\alpha}(u), X_{-\beta}(v_1)], \dots, X_{-\beta}(v_j)] = \prod_{\substack{\gamma \in [\alpha, \underbrace{-\beta, \dots, -\beta}_{j \text{ times}}]} X_{\gamma}(f_{\gamma}),$$

where $u \in M_{\alpha} = IV_{\alpha}$, and $v_1, \dots, v_j \in V_{-\beta}$. By the generalized Chevalley commutator formula $f_{\gamma} \in IV_{\gamma}$ for any γ . One readily sees that

$$[\alpha, \underbrace{-\beta, \dots, -\beta}_{j \text{ times}}] \subseteq \{\alpha - k\beta \mid k \geq j\} \cup \{i\alpha - l\beta \mid i \geq 2, l \geq 1\}.$$

By the hypothesis of the second induction, we have $M_{\gamma} = IV_{\gamma}$ for any $\gamma = \alpha - k\beta$ with $k > j$; hence $X_{\gamma}(f_{\gamma}) \in H$ for such γ .

Consider $\gamma = i\alpha - l\beta$, $i \geq 2, l \geq 1$. Let $m \in \mathbb{Z}$ be maximal such that $i\alpha + m\beta \in \Phi_P$ (we may have $m = -l$). We claim that $ht(i\alpha + m\beta) > ht(\alpha)$. Indeed, since $\alpha + \beta \notin \Phi_P$, by Lemma 2 (ii) we have $(i\alpha + m\beta) - \alpha \in \Phi_P$. Since α is a positive relative root linearly independent with β , we have $(i-1)\alpha + m\beta \in \Phi_P^+$, and hence $ht(i\alpha + m\beta) > ht(\alpha)$. Applying the induction hypothesis of the first induction to $i\alpha - l\beta = (i\alpha + m\beta) - (l+m)\beta$, we conclude that $X_{i\alpha-l\beta}(f_{i\alpha-l\beta}) \in H$ as well.

Summing up, we see that $X_{\gamma}(f_{\gamma}) \in H$ for any $\gamma \in [\alpha, \underbrace{-\beta, \dots, -\beta}_{j \text{ times}}]$ such that $\gamma \neq \alpha - j\beta$. Since $[\dots [X_{\alpha}(u), X_{-\beta}(v_1)], \dots, X_{-\beta}(v_j)] \in H$, this implies that $X_{\alpha-j\beta}(f_{\alpha-j\beta}) \in H$ too.

On the other hand, one readily sees that

$$f_{\alpha-j\beta} = N_{\alpha, \underbrace{-\beta, \dots, -\beta}_{j \text{ times}}}(u, v_1, \dots, v_j).$$

Since the structure constants of Φ are invertible in R , the polylinear map

$$N_{\alpha, \underbrace{-\beta, \dots, -\beta}_{j \text{ times}}} : V_{\alpha} \times V_{-\beta} \times \dots \times V_{-\beta} \rightarrow V_{\alpha-j\beta}$$

is surjective. Therefore,

$$N_{\alpha, \underbrace{-\beta, \dots, -\beta}_{j \text{ times}}}(IV_{\alpha}, V_{-\beta}, \dots, V_{-\beta}) = IV_{\alpha-j\beta}.$$

Hence $f_{\alpha-j\beta}$ runs over all elements of $IV_{\alpha-j\beta}$. This means that $X_{\alpha-j\beta}(IV_{\alpha-j\beta}) \subseteq H$, and $IV_{\alpha-j\beta} \subseteq M_{\alpha-j\beta}$. Therefore, $IV_{\alpha-j\beta} = M_{\alpha-j\beta}$. This settles the induction step of the second induction, and hence also that of the first induction. \square

4. CENTRALITY OF K_2 OVER LOCAL RINGS

4.1. Steinberg group. Let R be any commutative ring. Let G be an isotropic reductive group over R , $P, P^- \subseteq G$ two opposite parabolic subgroups of G , $L_P = P \cap P^-$ their common Levi subgroup. We assume that everything is of constant type over R , Φ_P and the relative root subschemes are well-defined over R , and $\text{rank } \Phi_P \geq 2$.

Definition 5. Let R' be an R -algebra, then the Steinberg group associated to G over R' is the group $St_P(R')$, generated as an abstract group by elements $\tilde{X}_{\alpha}(u)$, $\alpha \in \Phi_P$, $u \in V_{\alpha} \otimes_R R'$, subject to the relations

$$(5) \quad \tilde{X}_{\alpha}(v)\tilde{X}_{\alpha}(w) = \tilde{X}_{\alpha}(v+w) \prod_{i>1} \tilde{X}_{i\alpha}(q_{\alpha}^i(v, w)) \text{ for all } \alpha \in \Phi_P, v, w \in V_{\alpha} \otimes_R R',$$

and

$$(6) \quad [\tilde{X}_{\alpha}(u), \tilde{X}_{\beta}(v)] = \prod_{i,j>0} \tilde{X}_{i\alpha+j\beta}(N_{\alpha\beta ij}(u, v)), \text{ for all } \alpha, \beta \in \Phi_P \text{ such that } m\alpha \neq -k\beta \text{ for any } m, k > 0, \text{ and all } u \in V_{\alpha} \otimes_R R', v \in V_{\beta} \otimes_R R'.$$

Remark. Note that one can equivalently define $St_P(R')$ as the group generated by subgroups $\tilde{U}_{(\alpha)}(R') \cong U_{(\alpha)}(R')$, for all non-divisible $\alpha \in \Phi_P$, subject to the relations of the second type only, where the injections $\tilde{X}_{\alpha} : V_{\alpha} \otimes_R R' \rightarrow \tilde{U}_{(\alpha)}(R')$ are defined naturally.

One readily sees that $St_P(-)$ is a functor on the category of commutative R -algebras, since the root subschemes and the relations are functorial. Each group $St_P(R')$ is equipped with the natural surjective homomorphism

$$s_P = s_P(R') : St_P(R') \rightarrow E(R'),$$

which produces a natural transformation of functors.

For any subgroup of $E(R)$ generated by a set of relative root elements $X_{\alpha}(v)$, we will denote the subgroup of $St(R)$ generated by the respective liftings $\tilde{X}_{\alpha}(v)$ by the same letter or combination of letters, but with \sim on top. In particular, for any set $S \subseteq \Phi_P$ and any ideal $I \subseteq R$ we have

$$\tilde{U}_S(I) = \left\langle \tilde{X}_{\alpha}(V_{\alpha} \otimes_R I), \alpha \in S \right\rangle \leq St_P(R);$$

so that $\tilde{U}_{P^{\pm}}(I) = \tilde{U}_{\Phi_P^{\pm}}(I)$, and

$$\tilde{E}_{\alpha}(R) = \tilde{U}_{\mathbb{Z}\alpha \cap \Phi_P}(R) = \left\langle \tilde{U}_{(\alpha)}(R), \tilde{U}_{(-\alpha)}(R) \right\rangle.$$

Lemma 12. *Let $Q \leq P$ be two strictly proper parabolic subgroups of G over R , and $Q^- \subseteq P^-$ be opposite parabolic subgroups, such that Φ_P , Φ_Q and the respective relative root subschemes are well-defined over R .*

(i) *The inclusions $U_{P^\pm} \subseteq U_{Q^\pm}$ induce a surjective homomorphism*

$$\phi_{PQ} = \phi_{PQ}(R) : St_P(R) \twoheadrightarrow St_Q(R).$$

(ii) *For any relative root $\alpha \in \Phi_Q$ there is a relative root $\alpha' \in \Phi_P$ such that $\tilde{E}_\alpha(R) \leq \phi_{PQ}(\tilde{E}_{\alpha'}(R))$.*

(iii) *There is a positive integer $M > 0$, independent of the ring R , such that each element $\tilde{X}_\alpha(u)$, $\alpha \in \Phi_Q$, $u \in V_\alpha$, is a product of $\leq M$ elements of the form $\phi_{PQ}(\tilde{X}_\beta(v))$, $\beta \in \Phi_P \cap \mathbb{Z}\alpha'$, $v \in V_\beta$.*

Proof. This is proved exactly as [PSt1, Lemma 12]. Namely, one can establish the following statement using the Chevalley commutator formula: there exists $k > 0$ depending only on $\text{rank } \Phi_Q$, such that for any relative root $\alpha \in \Phi_Q$ and any $v \in V_\alpha$ there exist relative roots $\beta_i, \gamma_{ij} \in \Phi_P$, elements $v_i \in V_{\beta_i}$, $u_{ij} \in V_{\gamma_{ij}}$, and integers $k_i, n_i, l_{ij} > 0$ ($1 \leq i \leq m$, $1 \leq j \leq m_j$), one has the equality

$$(7) \quad X_\alpha(\xi \eta^k v) = \prod_{i=1}^m X_{\beta_i}(\xi^{k_i} \eta^{n_i} v_i) \prod_{j=1}^{m_i} X_{\gamma_{ij}}(\eta^{l_{ij}} u_{ij}),$$

where ξ, η are two commuting free variables over R .

In particular, the claim (ii) for $\alpha \in \pi_Q(\Phi \setminus \pi_P^{-1}(\Phi_P))$ is established by applying the Steinberg group version of this formula to the natural reductive subgroup of G of type $(\Pi \setminus J) \cup S \cup \Gamma b$, where $\Phi_Q = \Phi_{J, \Gamma}$ and $S \subseteq J \subseteq \Pi$ is the set of roots in J , which occur with non-zero coefficients in the decomposition of roots in $\pi_Q^{-1}(\alpha)$; b is the nearest to S simple root in J' , where $\Phi_P = \Phi_{J', \Gamma}$ (note that $J' \subseteq J$).

The claim (iii) is implied by the following observations. Suppose we have written down a factorization of the form (7) for all $\alpha \in \Phi_Q$ and $v = e \in V_\alpha$ running over a fixed system of generators of the finitely generated projective R -module V_α ; let N be a common upper bound for lengths of these factorizations. Then for any e as above and $\lambda \in R$ a factorization for $\tilde{X}_\alpha(\lambda e)$ is obtained by substituting $\xi = \lambda$ and $\eta = 1$ in (7), and hence has length $\leq N$. Now a factorization of $\tilde{X}_\alpha(v)$ for an arbitrary $v \in V_\alpha$ is obtained by decomposing v into a sum of products λe and using the formula (1); one readily sees that its length is bounded by $N \cdot \sum_{i \geq 1} \text{rank}_R V_{i\alpha} \leq N|\Phi|$. □

Lemma 13. *Let $S \subseteq \Phi_P$ be a set of relative roots closed under positive linear combinations, and such that for any $\alpha, \beta \in S$, $m\alpha \neq -k\beta$ for all $m, k > 0$. Then $s_P|_{\tilde{U}_S(R)} : \tilde{U}_S(R) \rightarrow U_S(R)$ is a group isomorphism. In particular, $s_P|_{\tilde{X}_\alpha(V_\alpha)} : \tilde{X}_\alpha(V_\alpha) \rightarrow X_\alpha(V_\alpha)$ is a bijection.*

Proof. This is proved exactly as [Deo, Lemma 1.10]. □

Let $\alpha \in \Phi_P$ be a relative root, recall that by Lemma 1 we have

$$\Phi_P \cap \mathbb{Z}\alpha = \{\pm\alpha, \pm 2\alpha, \dots, \pm m_\alpha\alpha\}.$$

For $a \in \tilde{E}_\alpha(A)$, $u_i \in V_{i\alpha}$, $1 \leq i \leq m_\alpha$, we define

$$\tilde{Z}_\alpha(a, u_1, \dots, u_{m_\alpha}) = a \left(\prod_{i=1}^{m_\alpha} \tilde{X}_{i\alpha}(u_i) \right) a^{-1}.$$

The elements $Z_\alpha(a, u_1, \dots, u_{m_\alpha}) \in E_P(R)$ are defined in the same way.

We set

$$\tilde{E}(R, I) = \left\langle \tilde{X}_\alpha(IV_\alpha), \alpha \in \Phi_P \right\rangle^{St_P(R)}.$$

Lemma 14. *Assume that $\text{rank } \Phi_P \geq 2$. Let I be an ideal of R . Then $\tilde{E}(R, I)$ is generated by all $\tilde{Z}_\alpha(a, u)$, $a \in \tilde{E}_\alpha(A)$, $u = (u_1, \dots, u_{m_\alpha})$, $u_i \in IV_{i\alpha}$, $1 \leq i \leq m_\alpha$.*

Proof. The corresponding statement for $E(R, I)$ was established in [St12, Lemma 4.3]. Since the proof used only the Chevalley commutator formula, the same statement holds for $\tilde{E}(R, I)$. \square

Lemma 15. *For any ideal $I \subseteq R$ there is a short exact sequence*

$$1 \longrightarrow \tilde{E}(R, I) \longrightarrow St_P(R) \xrightarrow{\tilde{\rho}} St_P(R/I) \longrightarrow 1,$$

where $\tilde{\rho}$ is induced by the residue map $\rho : R \rightarrow R/I$.

Proof. Since the maps $U_{(\alpha)}(R) \rightarrow U_{(\alpha)}(R/I)$ induced by ρ are surjective, the natural group homomorphism

$$\tilde{\rho} : St_P(R) \rightarrow St_P(R/I)$$

is surjective too. It remains to show that $\tilde{E}(R, I) = \ker \tilde{\rho}$. Clearly, $\tilde{E}(R, I) \subseteq \ker \tilde{\rho}$, so $\tilde{\rho}$ factors through the surjective homomorphism $\bar{\rho} : St_P(R)/\tilde{E}(R, I) \rightarrow St_P(R/I)$.

The group $St_P(R)/\tilde{E}(R, I)$ is generated by the subgroups $\tilde{U}_{(\alpha)}(R)/(\tilde{U}_{(\alpha)}(R) \cap \tilde{E}(R, I))$, and one readily shows by applying $s_P(R)$ that

$$\tilde{U}_{(\alpha)}(R)/(\tilde{U}_{(\alpha)}(R) \cap \tilde{E}(R, I)) \cong \tilde{U}_{(\alpha)}(R)/\tilde{U}_{(\alpha)}(I) \cong U_{(\alpha)}(R)/U_{(\alpha)}(I) \cong U_{(\alpha)}(R/I).$$

Since the Chevalley commutator formula is functorial with respect to $R \rightarrow R/I$, the subgroups $\tilde{U}_{(\alpha)}(R)/(\tilde{U}_{(\alpha)}(R) \cap \tilde{E}(R, I)) \cong U_{(\alpha)}(R/I)$ in $St_P(R)/\tilde{E}(R, I)$ also satisfy the same relations as the ones in $St_P(R/I)$. Therefore, by the universal property of $St_P(R/I)$ the canonical map $s_P : St_P(R/I) \rightarrow E(R/I)$ factors through the group $St_P(R)/\tilde{E}(R, I)$, producing an inverse for $\bar{\rho}$. \square

4.2. Big cell in $St_P(R)$. We keep the notation from the previous subsection.

Definition 6. *We define the group $\tilde{L}_P(R)$ as the subgroup of $St_P(R)$ generated by all elements \tilde{h} such that $\tilde{h} \in \tilde{E}_\alpha(R)$ for some $\alpha \in \Phi_P$, and $s_P(\tilde{h}) \in L_P(R)$.*

The group $\tilde{L}_P(I)$ is defined to be the subgroup of $\tilde{L}_P(R)$ generated by all elements $\tilde{h} = x_1 y_2 x_2 y_2$, where $x_1, x_2 \in \tilde{U}_{(\alpha)}(R)$ and $y_1, y_2 \in \tilde{U}_{(-\alpha)}(R)$, $s_P(\tilde{h}) \in L_P(R)$, and in addition $y_1 \in \tilde{U}_{(-\alpha)}(I)$ or $x_1 \in \tilde{U}_{(\alpha)}(I)$. Note that the last condition is equivalent to $y_2 \in \tilde{U}_{(-\alpha)}(I)$ or $x_2 \in \tilde{U}_{(\alpha)}(I)$ respectively, and implies $s_P(\tilde{h}) \in L_P(R, I)$.

Remark. We don't know in general if the map s_P restricted to $\tilde{L}_P(R)$ provides a surjection onto $L_P(R) \cap E(R)$.

Lemma 16. *Take $\tilde{h} \in \tilde{L}_P(R)$ and set $h = s_P(\tilde{h})$. Then for any $\alpha \in \Phi_P$, $v \in V_\alpha$, one has*

$$(8) \quad \tilde{h}\tilde{X}_\alpha(v)\tilde{h}^{-1} = (s_P|_{\tilde{U}_{(\alpha)}})^{-1}(hX_\alpha(v)h^{-1}).$$

Proof. It is enough to prove the claim for all $\tilde{h} \in \langle U_{(\beta)}(R), U_{(-\beta)}(R) \rangle$, $\beta \in \Phi_P$, such that $h = s_P(\tilde{h}) \in L_P(R)$. We have $hX_\alpha(v)h^{-1} \in U_{(\alpha)}(R)$.

Assume first that α and β are linearly independent. Then $S = (\mathbb{Z}\beta + \mathbb{N}\alpha) \cap \Phi_P$ is a set of roots that is subject to Lemma 13. It is easy to see that we can write \tilde{h} as a product of elements of the form $\tilde{Z}_\beta(b, u_1, \dots, u_{m_\beta})$, $b \in \tilde{E}_\beta(R)$, $u_i \in V_{i\beta}$, $1 \leq i \leq m_\beta$. Then [St12, Lemma 4.4] implies that $\tilde{h}\tilde{X}_\alpha(v)\tilde{h}^{-1} \in \tilde{U}_S(R)$. Then Lemma 13 implies the equality (8).

Assume now that α and β are linearly dependent. By [PSt1, Lemma 11], which treated the case of relative root elements in $E(R)$ but used only the Chevalley commutator formula that holds in $St_P(R)$, there are relative roots $\alpha_1, \dots, \alpha_n$ in Φ_P , linearly independent with α , and elements $v_i \in V_{\alpha_i}$, such that

$$\tilde{X}_\alpha(v) = \prod_{i=1}^n \tilde{X}_{\alpha_i}(v_i).$$

This reduces the proof to the previous case. \square

Lemma 17. *(i) If for some $x, x' \in \tilde{U}_P(R)$, $y, y' \in \tilde{U}_{P^-}(R)$ and $t, t' \in \tilde{L}_P(R)$ one has $s_P(xty) = s_P(x't'y')$, then $x = x'$, $y = y'$.*

(ii) The natural map $\tilde{U}_P(R) \times \tilde{L}_P(R) \times \tilde{U}_{P^-}(R) \rightarrow St_P(R)$ induced by multiplication in $St_P(R)$ is injective.

Proof. The natural map $U_P(R) \times L_P(R) \times U_{P^-}(R) \rightarrow G(R)$ is injective, hence $s_P(xty) = s_P(x't'y')$ implies $s_P(x) = s_P(x')$ and $s_P(y) = s_P(y')$. By Lemma 13 this implies $x = x'$ and $y = y'$, which proves (i). The claim (ii) follows by applying s_P to the image of the map under consideration. \square

Lemma 18. *In the setting of Lemma 12, one has*

$$\ker \phi_{PQ} \subseteq \tilde{L}_P(R) \cap \text{Cent}(St_P(R))$$

and $\phi_{PQ}^{-1}(\tilde{L}_Q(R)) = \tilde{L}_P(R)$.

Proof. Let $\pi_{PQ} : \Phi_Q \rightarrow \Phi_P \cup \{0\}$ be the natural surjection induced by the inclusion between the types of P and Q . By Lemma 12 for any $\alpha \in \Phi_Q$ and $u \in V_\alpha$ there is an element $\hat{X}_\alpha(u) \in St_P(R)$ such that $\phi_{PQ}(\hat{X}_\alpha(u)) = \tilde{X}_\alpha(u)$. Moreover, if $\pi_{P,Q}(\alpha) \neq 0$, then the definition of ϕ_{PQ} allows to stipulate that $\hat{X}_\alpha(u) \in \tilde{U}_{(\alpha')}(R)$, where $\alpha' = \pi_{PQ}(\alpha)$.

On the other hand, for any $\alpha' \in \Phi_P$ the set $\pi_Q(\pi_P^{-1}(\alpha')) \subseteq \Phi_Q$ is, clearly, a unipotent closed subset, therefore, applying Lemma 13, we see that $\tilde{X}_{\alpha'}(V_{\alpha'})$ is contained in the subgroup of $St_P(R)$ generated by elements of the form $\hat{X}_\alpha(u)$, $\alpha \in \Phi_Q$, $u \in V_\alpha$. Therefore, the free group generated by all $X_\alpha(u)$, $\alpha \in \Phi_Q$, $u \in V_\alpha$, surjects onto $St_P(R)$. By the definition of $St_Q(R)$, this implies that \ker_{PQ} is contained in the subgroup of

$St_P(R)$ generated by the expressions of the form

(9)

$$\sigma(\alpha, v, w) = \hat{X}_\alpha(v) \hat{X}_\alpha(w) \left(\hat{X}_\alpha(v+w) \prod_{i>1} \hat{X}_{i\alpha}(q_\alpha^i(v, w)) \right)^{-1} \text{ for all } \alpha \in \Phi_Q, v, w \in V_\alpha \otimes_R R',$$

and

(10)

$$\tau(\alpha, \beta, u, v) = [\hat{X}_\alpha(u), \hat{X}_\beta(v)] \left(\prod_{i,j>0} \hat{X}_{i\alpha+j\beta}(N_{\alpha\beta ij}(u, v)) \right)^{-1}, \text{ for all } \alpha, \beta \in \Phi_Q \text{ such that } m\alpha \neq -k\beta \text{ for any } m, k > 0, \text{ and all } u \in V_\alpha \otimes_R R', v \in V_\beta \otimes_R R'.$$

We will show that all this expressions are contained in $\tilde{L}_P(R)$.

Since $s_P = s_Q \circ \phi_{PQ}$, we have $s_P(\sigma(\alpha, v, w)) = s_P(\tau(\alpha, \beta, u, v)) = 1$ for all possible α, β, u, v, w . Therefore, it is enough to find $\alpha' \in \Phi_P$ such that $\sigma(\alpha, v, w)$, or, respectively, $\tau(\alpha, \beta, u, v)$ belongs to $\tilde{E}_{\alpha'}(R)$. By Lemma 12 (ii) this holds for all elements $\sigma(\alpha, v, w)$. It is also clear for all $\tau(\alpha, \beta, u, v)$ such that $\pi_{PQ}(\alpha) \neq 0$ and $\pi_{PQ}(\beta) \neq 0$ are collinear elements of Φ_P . If $\pi_{PQ}(\alpha) \neq 0$ and $\pi_{PQ}(\beta) \neq 0$ are non-collinear elements of Φ_P , then they span a unipotent closed subset of Φ_P , and by applying Lemma 13, we readily deduce that $\tau(\alpha, \beta, u, v) = 1$.

It remains to consider the elements $\tau(\alpha, \beta, u, v)$, where $\alpha \in \Phi_Q$ satisfies $\pi_{PQ}(\alpha) = 0$. Note that this implies $s_Q(\tilde{X}_\alpha(u)) \in L_P(R)$, and hence $s_P(\hat{X}_\alpha(u))$. Therefore, Lemma 12 (ii) implies $\hat{X}_\alpha(u) \in \tilde{L}_P(R)$. Now if $\pi_{PQ}(\beta) = 0$ too, the same argument shows that all factors in $\tau(\alpha, \beta, u, v)$ belong to $\tilde{L}_P(R)$, hence $\tau(\alpha, \beta, u, v) \in \tilde{L}_P(R)$. If $\pi_{PQ}(\beta) \neq 0$, then we have $\hat{X}_{i\alpha+j\beta}(V_{i\alpha+j\beta}) \subseteq \tilde{U}_{(\beta')}(R)$ for all $i, j \in \mathbb{Z}$ with $j > 0$, where $\beta' = \pi_{PQ}(\beta)$. Then Lemma 13 combined with Lemma 16 implies that $\tau(\alpha, \beta, u, v) = 1$.

We have proved that $\ker \phi_{PQ} \subseteq \tilde{L}_P(R)$. Combining Lemma 13 with Lemma 16 again, we see that $\ker \phi_{PQ} \subseteq \text{Cent}(St_P(R))$, since $s_P(\ker \phi_{PQ}) = 1$.

To finish the proof of the lemma, it remains to note that since $L_Q(R) \subseteq L_P(R)$, Lemma 12 (ii) implies that any standard generator of $\tilde{L}_Q(R)$ has a lifting in $\tilde{L}_P(R)$. \square

4.3. Centrality of K_2 over a local ring. Throughout this section, R is a local ring, $I \subseteq R$ is its maximal ideal, $\rho : R \rightarrow R/I = l$ is the residue map.

Let G be an isotropic simply connected simple reductive group over R , $P, P^- \subseteq G$ two opposite parabolic subgroups of G , $L_P = P \cap P^-$ their common Levi subgroup. We assume that $\text{rank } \Phi_P \geq 2$.

Lemma 19. *One has $\tilde{E}(R, I) \subseteq \tilde{U}_P(I) \tilde{L}_P(R) \tilde{U}_{P^-}(I)$.*

Proof. Set $Z = \tilde{U}_P(I) \tilde{L}_P(R) \tilde{U}_{P^-}(I)$. By Lemma 14 the group $\tilde{E}(R, I)$ is generated by elements $\tilde{Z}_\alpha(a, u_1, \dots, u_{m_\alpha})$, $\alpha \in \Phi_P$, $a \in \tilde{E}_\alpha(R)$, $u_i \in IV_{i\alpha}$. Hence it is enough to show that $\tilde{Z}_\alpha(a, u_1, \dots, u_{m_\alpha})Z \subseteq Z$.

Let $G_\alpha \leq G$ be the reductive subgroup corresponding to α . Then $L_P \leq G_\alpha$ is the common Levi subgroup of two opposite parabolic subgroups $L_P U_{(\alpha)}$ and $L_P U_{(-\alpha)}$ of G_α , and $U_{(\alpha)} L_P U_{(-\alpha)}$ is an open subscheme (the big cell) of G_α . Then, since

$$\rho \left(s_P(\tilde{Z}_\alpha(a, u_1, \dots, u_{m_\alpha})) \right) = 1,$$

we have

$$s_P(\tilde{Z}_\alpha(a, u_1, \dots, u_{m_\alpha})) \in U_{(\alpha)}(I)L_P(R, I)U_{(-\alpha)}(I).$$

Therefore, by the definition of $\tilde{L}_P(R)$, we have

$$\tilde{Z}_\alpha(a, u_1, \dots, u_{m_\alpha}) \in \tilde{U}_{(\alpha)}(I)\tilde{L}_P(R)\tilde{U}_{(-\alpha)}(I).$$

By Lemma 16 we have $\tilde{L}_P(R)Z \subseteq Z$. Therefore, in order to establish the inclusion $\tilde{Z}_\alpha(a, u_1, \dots, u_{m_\alpha})Z \subseteq Z$, it is enough to show that

$$(11) \quad \tilde{U}_{(-\alpha)}(I)\tilde{U}_P(I) \subseteq Z$$

for any non-divisible relative root $\alpha \in \Phi_P^+$. One shows exactly as above that

$$(12) \quad \tilde{U}_{(-\alpha)}(I)\tilde{U}_{(\alpha)}(I) \subseteq U_{(\alpha)}(I)\tilde{L}_P(R)U_{(-\tilde{\alpha})}(I).$$

Let $\beta \in \Phi_P^+$ be non-collinear to α . By the Chevalley commutator formula one has

$$(13) \quad \tilde{U}_{(-\alpha)}(I)\tilde{U}_{(\beta)}(I) \subseteq \prod_{\substack{i \geq 0, j > 0, \\ -i\alpha + j\beta \in \Phi_P^+}} \tilde{U}_{(-i\alpha + j\beta)}(I) \cdot \prod_{\substack{i > 0, j \geq 0, \\ -i\alpha + j\beta \in \Phi_P^-}} \tilde{U}_{(-i\alpha + j\beta)}(I).$$

By Lemma 13 the group $\tilde{U}_P(I)$ can be written as a product of $\tilde{U}_{(\beta)}(I)$ with β running over all non-divisible elements of Φ_P^+ in any fixed order. Then formulas (12) and (13) together imply (11). \square

Lemma 20. *The map $\tilde{\rho}|_{\tilde{L}_P(R)} : \tilde{L}_P(R) \rightarrow \tilde{L}_P(l)$ is surjective.*

Proof. Let $\tilde{h} \in St_P(l)$ be a standard generator of $\tilde{L}_P(l)$, i.e. $\tilde{h} \in \tilde{E}_\alpha(l)$, $\alpha \in \Phi_P$, and $h = s_P(l)(\tilde{h}) \in L_P(l)$. Let $G_\alpha \leq G$ be the reductive subgroup corresponding to α . Then $L_P \leq G_\alpha$ is the common Levi subgroup of two opposite parabolic subgroups $L_P U_{(\alpha)}$ and $L_P U_{(-\alpha)}$ of G_α , and $U_{(\alpha)} L_P U_{(-\alpha)}$ is an open subscheme (the big cell) of G_α . Then, since $h \in L_P(l)$, we have

$$s_P(R)(\tilde{\rho}^{-1}(\tilde{h})) \subseteq U_{(\alpha)}(I)L_P(R)U_{(-\alpha)}(I).$$

Since $\tilde{h} \in \tilde{E}_\alpha(l)$, it has a preimage in $\tilde{E}_\alpha(R)$. Multiplying this preimage, if necessary, by the corresponding elements in $\tilde{U}_{(\pm\alpha)}(I)$, we obtain an element in $\tilde{L}_P(R)$. \square

The following lemma is established using the results of V. V. Deodhar [Deo] concerning the covering \tilde{G} of the subgroup $G_l(l)^+$ of $G_l(l)$, which is a particular case of the Steinberg group in our sense, namely the one associated to a minimal parabolic subgroup of G_l .

Lemma 21. *Assume that $g \in St_P(R)$ is such that $s_P(g) = 1$. Then $g \in \tilde{L}_P(R)\tilde{E}_P(R, I)$.*

Proof. Let Q be a minimal parabolic subgroup of G_l contained in P_l , L_Q a Levi subgroup of Q contained in $(L_P)_l$. The system of relative roots Φ_Q and the subgroups $U_{(\alpha)}$, $\alpha \in \Phi_Q$, in the sense of [PSt1], are identified in this case with the relative root system of G_l and respective root subgroups in the sense of [BT], and, accordingly, in [Deo]. This readily follows by descent from a field extension that splits G_l . Since $\text{rank } \Phi_P \geq 2$, the results of [PSt1] imply

$$E_P(l) = E_{P_l}(l) = E_Q(l) = G_l(l)^+,$$

see [St12, Theorem 2.1] for a detailed proof.

The definition [Deo, 1.9] provides a covering group \widetilde{G}_l of $G_l(l)^+ = E_Q(l)$, which is precisely $St_Q(l)$ in our notation.

We will make use of the following commutative diagram consisting of surjective group homomorphisms:

$$(14) \quad \begin{array}{ccccc} St_P(R) & \xrightarrow{\tilde{\rho}} & St_P(R)/\tilde{E}_P(R, I) \cong St_P(R/I) & \xrightarrow{\phi_{PQ}} & St_Q(R/I) \\ \downarrow s_P & & \searrow s_P & & \downarrow s_Q \\ E(R) = E_P(R) & \xrightarrow{\rho} & E_P(R/I) = E_Q(R/I) & & \end{array}$$

Here the map ϕ_{PQ} is the one constructed in Lemma 12, the map $\tilde{\rho}$ is the one from Lemma 15.

Deodhar shows in [Deo, Prop. 1.16] that $St_Q(R/I) = St_Q(l)$ is a central extension of $E_Q(R/I) = E_Q(l)$, and moreover $\ker s_Q \subseteq \tilde{H}$, where \tilde{H} is a certain subgroup of $St_Q(l)$. Note that this result does not use the assumption $|l| \geq 16$ present in [Deo, Theorem 1.9]; that assumption was required to prove that the central extension is universal. By definition, the subgroup \tilde{H} is generated by some elements $\tilde{h}_\alpha(u, u')$, where $\alpha \in \Phi_Q$, $1 \neq u, u' \in \tilde{U}_{(\alpha)}(l)$, such that $\tilde{h}_\alpha(u, u') \in \tilde{U}_{(\alpha)}(l)$ and $s_Q(\tilde{h}(u, u')) \in L_Q(l)$. This implies that $\ker s_Q \subseteq \tilde{L}_Q(l)$.

Let $g \in St_P(R)$ be such that $s_P(g) = 1$. Then $\phi_{PQ}(\tilde{\rho}(g)) \in \ker s_Q \subseteq \tilde{L}_Q(l)$. By Lemma 18 this implies that $\tilde{\rho}(g) \in \tilde{L}_P(R)$. Since $\ker \tilde{\rho} = \tilde{E}(R, I)$ by Lemma 15, by Lemma 20 we have $g \in \tilde{L}_P(R)\tilde{E}(R, I)$. \square

Proof of Theorem 3. By Lemma 21 combined with Lemmas 16 and 19 we have

$$\ker(s_P : St_P(R) \rightarrow E(R)) \subseteq \tilde{U}_P(R)\tilde{L}_P(R)\tilde{U}_{P^-}(R).$$

By Lemma 17 this implies that actually

$$\ker(s_P : St_P(R) \rightarrow E(R)) \subseteq \tilde{L}_P(R).$$

Applying Lemma 16 and Lemma 13, we conclude that $\ker(s_P : St_P(R) \rightarrow E(R))$ is contained in the center of $St_P(R)$. \square

5. PROOF OF THE MAIN THEOREM

In this section we establish Theorem 1. Actually, we show that the proof of the corresponding statement for split groups G given in [RR] carries over to the isotropic case almost literally, once we have at our disposal Theorems 2 and 3.

5.1. Bounded generation over profinite completion. The proof of the main result of [RR] at the final step uses bounded generation of the elementary subgroup of a Chevalley group over the profinite completion \hat{R} of a commutative ring R with respect to elementary root generators. To settle the more general case we consider, we will need the following lemma.

Lemma 22. *Let R be a local ring, G a quasi-split simple simply connected group over R , P a parabolic subgroup of G . Then there is an integer $N > 0$ such that each element of $E_P(R)$ is a product of $\leq N$ elements of $U_P(R)$ and $U_{P^-}(R)$.*

Proof. Let B be a Borel subgroup of G contained in P . If $\text{rank } \Phi_P = 1$, then we have $B = P$; otherwise Lemma 12 (iii) shows that the claim of the present lemma for P is implied by the one for B . Thus we can assume $P = B$.

If G is split, the claim follows from [RR, Proposition 2.2]. If G is quasi-split but not split, then G is of outer type 2A_n , $n \geq 2$; 2D_n , $n \geq 4$; ${}^3(6)D_4$ or 2E_6 . Let T be a maximal torus of G contained in B . The Gauss decomposition in $G(R)$ [SGA3, Exp. XXVI, Théorème 5.1] states that

$$G(R) = U_B(R)U_{B^-}(R)T(R)U_B(R).$$

Therefore, it is enough to show that $T(B)$ is boundedly generated by the elements of $U_B(R)$ and $U_{B^-}(R)$. In what follows we use the terminology and notation of [PSt2] and [SGA3, Exp. XXVI]. Let $R \rightarrow S$ is a connected Galois ring extension splitting the Dynkin scheme $\text{Dyn}(G)$ of G . It is easy to see (e.g. [PSt2, Proposition 1]) that $T(R)$ is a product of maximal tori of the standard subgroups H of G of the form $\text{SL}_{2,R}$, $\text{SU}_{3,S/R}$ or $R_{S/R}(\text{SL}_{2,S})$, corresponding to the distinct $\text{Gal}(S/R)$ -orbits of $\text{Dyn}(G)$, or, in other words, to the simple relative roots $\alpha = \alpha(H) \in \Phi_B$. Here $R_{S/R}(\text{SL}_{2,S})$ denotes the Weil restriction of $\text{SL}_{2,S}$ from S to R , and $\text{SU}_{3,S/R}$ stands for the group of type 2A_2 split by the extension S/R . For all three types of groups one readily sees that $H(R)$ is boundedly generated by the R -points of the unipotent radicals $U_{(\pm\alpha)}(R)$ of its opposite parabolic subgroups $B^\pm \cap H$. The corresponding formulas for $\text{SU}_{3,S/R}$ can be found, for example, in [Abe1, p. 196]. \square

5.2. End of the proof.

Proof of Theorem 1. The proof almost literally repeats that in [RR]. We reproduce it for the sake of completeness. Whenever possible, we use the same notation as [RR]. The only essential change in notation is that we denote by \hat{R}_m the m -adic completion of R at a maximal ideal m , instead of R_m there, to avoid confusion with the localization of R at m . We also denote by

$$p : \widehat{E(R)} \rightarrow \overline{E(R)}$$

the natural homomorphism.

By [RR, Lemma 2.1] there is an isomorphism of topological rings

$$\hat{R} = \prod_m \hat{R}_m,$$

where m runs over all maximal ideals of R of finite index. Each ring \hat{R}_m is a local complete ring with a finite residue field, hence G is quasi-split over \hat{R}_m by Lang's theorem. Then Lemma 22 implies that $G(\hat{R}) = E(\hat{R})$ is boundedly generated by $X_\alpha(v)$, $\alpha \in \Phi_P$, $v \in V_\alpha \otimes_R \hat{R}$. One shows exactly as in [RR, Prop. 2.5] that

$$\overline{E(R)} = E(\hat{R}) = \left\langle X_\alpha(V_\alpha \otimes_R \hat{R}), \alpha \in \Phi_P \right\rangle.$$

By Theorem 2 the profinite and congruence topologies coincide on $X_\alpha(V_\alpha) \subseteq E(R)$ for any $\alpha \in \Phi_P$. Therefore, $\widehat{E(R)}$ contains a system of subsets $\hat{X}_\alpha(V_\alpha \otimes_R \hat{R})$, $\alpha \in \Phi_P$, such that $p|_{\hat{X}_\alpha(V_\alpha \otimes_R \hat{R})}$ is a homomorphism onto $X_\alpha(V_\alpha \otimes_R \hat{R}) \subseteq E(\hat{R})$. Since the polynomial maps q_α^i and $N_{\alpha,\beta,i,j}$ in the relations (1) and (2) are continuous in the

congruence topology, the subsets $\hat{X}_\alpha(V_\alpha \otimes_R \hat{R})$, $\alpha \in \Phi_P$, satisfy the same relations (1) and (2) as the corresponding root subsets in $E(\hat{R})$.

Denote by $\hat{\Gamma}_m$ (respectively, $\hat{\Gamma}'_m$) the subgroup of $\widehat{E(R)}$ generated by all $\hat{X}_\alpha(V_\alpha \otimes_R \hat{R}_m)$, $\alpha \in \Phi_P$ (respectively, by all $\hat{X}_\alpha(V_\alpha \otimes_R \prod_{n \neq m} \hat{R}_n)$). The relations (2) together with [PSt1, Lemma 11] readily imply that the subgroups $\hat{\Gamma}_m$ and $\hat{\Gamma}'_m$ centralize each other in $\widehat{E(R)}$.

By Theorem 3 the kernel of the restriction $p|_{\hat{\Gamma}_m}$ lies in the center of $\hat{\Gamma}_m$ for any maximal ideal of finite index m in R . Since $\hat{\Gamma}_m$ centralizes $\hat{\Gamma}'_m$, we conclude that $\hat{\Gamma}_m$ centralizes the intersection $\mathcal{C}_E(R) \cap \Delta_m$, where $\Delta_m = \hat{\Gamma}_m \hat{\Gamma}'_m$.

One concludes as in [RR, Lemma 4.4] that Δ_m , as well as the subgroup $\Delta \leq \widehat{E(R)}$ generated by all $\hat{\Gamma}_m$, are both dense in $\widehat{E(R)}$. Since $E(\hat{R})$ is boundedly generated by $X_\alpha(V_\alpha \otimes_R \hat{R})$, $\alpha \in \Phi_P$, there is a finite sequence of relative roots $\alpha_1, \dots, \alpha_N$ such that the map

$$\prod_{i=1}^N (V_{\alpha_i} \otimes_R \hat{R}) \rightarrow E(\hat{R}), \quad (v_i) \mapsto \prod X_{\alpha_i}(v_i),$$

is surjective. This gives a compact subset

$$\prod_{i=1}^N \hat{X}_{\alpha_i}(V_{\alpha_i} \otimes_R \hat{R}) \subseteq \Delta_m \subseteq \widehat{E(R)}$$

such that its image under p is the whole of $E(\hat{R})$. Applying [RR, Lemma 4.3], we conclude that $\mathcal{C}_E(R) \cap \Delta_m$ is dense in $\mathcal{C}_E(R)$.

Therefore, $\hat{\Gamma}_m$ centralizes $\mathcal{C}_E(R)$ for any m . Since these subgroups generate the dense subgroup Δ in $\widehat{E(R)}$, we deduce that $\widehat{E(R)}$ centralizes $\mathcal{C}_E(R)$, as required. \square

Proof of Corollary 1. By [St12, Corollary 1.6] and [St12, Theorem 1.3], we have

$$G(\mathbb{F}_q[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]) = G(\mathbb{F}_q)E(\mathbb{F}_q[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}])$$

and

$$G(A[X_1, \dots, X_n]) = G(A)E(A[X_1, \dots, X_n]).$$

By [SGA3, Exp. XXVI, Th. 5.1] we have $K_1^G(A) = K_1^G(\mathbb{F}_q) = 1$, hence the claim. \square

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